

# Two-Body systems from $SL(2, \mathbb{C})$ -tops

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It is shown that  $sl(2, \mathbb{C})$  Euler-Arnold tops are equivalent to the two-body systems of Calogero-Moser type. We prove that generic Hamiltonians of  $sl(2, \mathbb{C})$  tops are equivalent to one of three canonical Hamiltonians. For all canonical Hamiltonians the corresponding two-body system is found. Bosonisation formulas for each case are obtained explicitly. Relations with Antonov-Zabrodin-Hasegawa R-matrix are discussed.

## 1 Introduction

A most famous examples of  $sl(2, \mathbb{C})$  Euler-Arnold tops are the so called  $XYZ$ ,  $XXZ$ ,  $XXX$  tops. They can be defined by their Lax operators. The Hamiltonian of the system is identified with free term in the expansion of  $Tr(L(z)^2)$  into series with respect to spectral parameter  $z$ . The Lax operators  $L(z)$  for these systems can be found from the equation for the linear Poisson brackets:

$$\{L(z) \otimes 1, 1 \otimes L(z)\} = [r, 1 \otimes L(z) + L(z) \otimes 1],$$

where  $r$  is the elliptical, the trigonometrical or the rational classical Baxter  $sl(2, \mathbb{C})$  r-matrix.

In the framework of the Hitchin approach to integrable systems [3], [4], [5], [6] the connection between the  $XYZ$  top, and the two-body Calogero-Moser system was established in [1]. It was shown that the Lax operators  $L_{ell}^{top}(z)$  and  $L_{ell}^{CM}(z)$  for systems are conjugated by the gauge transformation:

$$L_{ell}^{top}(z) = \Xi(z) L_{ell}^{CM}(z) \Xi^{-1}(z). \quad (1)$$

Here the matrix  $\Xi(z)$  is singular at the point  $z = 0$ . Let

$$M^{top} = \{S_j \quad j = 1, 2, 3 : S_1^2 + S_2^2 + S_3^2 = \nu^2\}$$

be a  $sl(2, \mathbb{C})$  coadjoint orbit. It is a phase space for the  $sl(2, \mathbb{C})$  Euler-Arnold tops with a Poisson bracket  $\{S_i, S_j\} = 2i \epsilon_{ijk} S_k$ . Define a coordinate  $q$  and a momentum  $p$  in the center mass of two-body Calogero-Moser system. The so called bosonisation formulas can be read off from (1). These formulas explicitly give the symplectomorphism between systems:

$$S_1 = -\frac{\theta_{10}(0)\theta_{10}(2q)}{\vartheta'(0)\vartheta(2q)}p + \frac{\theta_{10}^2(0)\theta_{00}(2q)\theta_{01}(2q)}{\theta_{00}(0)\theta_{01}(0)\vartheta^2(2q)}\nu, \quad (2)$$

$$S_2 = \frac{\theta_{00}(0)\theta_{00}(2q)}{i\vartheta'(0)\vartheta(2q)}p - \frac{\theta_{00}^2(0)\theta_{10}(2q)\theta_{01}(2q)}{i\theta_{10}(0)\theta_{01}(0)\vartheta^2(2q)}\nu, \quad (3)$$

$$S_3 = -\frac{\theta_{01}(0)\theta_{01}(2q)}{\vartheta'(0)\vartheta(2q)}p + \frac{\theta_{01}^2(0)\theta_{00}(2q)\theta_{10}(2q)}{\theta_{00}(0)\theta_{10}(0)\vartheta^2(2q)}\nu, \quad (4)$$

where  $\vartheta(q)$ ,  $\theta_{\alpha\beta}(q)$  are the standard elliptic theta function and the theta functions with characteristics respectively. Evidently, the connection between the tops and the Calogero-Moser systems should exist in trigonometrical and rational limits. However, in the naive trigonometric limit  $\tau \rightarrow i\infty$  the matrix elements of  $\Xi(z)$  are divergent. This difficulty was overcome in [8] by applying an additional gauge transformation  $A(\tau)$  depending on modular parameter  $\tau$  which is singular at point  $\tau = i\infty$ :

$$A(\tau) L_{ell}^{top}(z) A^{-1}(\tau) = A(\tau) \Xi L_{ell}^{CM}(z) \Xi^{-1} A^{-1}(\tau)$$

Upon taking the limits:

$$L_{trig}^{top}(z) = \lim_{\tau \rightarrow i\infty} A(\tau) L_{ell}^{top}(z) A^{-1}(\tau), \quad \Xi'(z) = \lim_{\tau \rightarrow i\infty} A(\tau) \Xi(z), \quad L_{trig}^{CM}(z) = \lim_{\tau \rightarrow i\infty} L_{ell}^{CM}(z),$$

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we get the regular trigonometric version of (1):

$$L_{trig}^{'top}(z) = \Xi'(z) L_{trig}^{CM}(z) \Xi'^{-1}(z). \quad (5)$$

Due to the fact that matrix  $A(\tau)$  is singular at the point  $\tau = i\infty$  this new trigonometric Lax operator is not gauge equivalent to the standard one:

$$L_{trig}^{top}(z) = \lim_{\tau \rightarrow i\infty} L_{ell}^{top}(z),$$

In this way we obtain a distinct from the ordinary  $XXZ$ -top Hamiltonian system on the coadjoint  $\mathfrak{sl}(2, \mathbb{C})$  orbit. Therefore, in addition to standard  $XXZ$  and  $XXX$  tops there exist nonequivalent to them trigonometrical and rational degenerations of elliptic  $XYZ$  top. It follows from (5) that these tops describe the evolution of the trigonometrical and the rational Calogero-Moser systems. In what follows we will denote these new tops as  $XXZ'$  and  $XXX'$  tops respectively.

Our approach to the problem is different from [8]. In section 3 we show that a general Hamiltonian of  $\mathfrak{sl}(2, \mathbb{C})$  top is gauge equivalent to one of three canonical Hamiltonian presented below. We obtain the bosonisation formulas for all top Hamiltonians in a simple way in section 4. By these means in sections 5-7 we receive the bosonisation formulas for each canonical Hamiltonian. We see that the bosonisations corresponding to the canonical Hamiltonians map tops to the two-body Calogero-Moser systems. Therefore presented canonical Hamiltonians are the Hamiltonians of  $XYZ$ ,  $XXZ'$  and  $XXX'$  tops. The bosonisation formulas obtained for  $XYZ$  top coincide with ones obtained in [1]. Formulas received for  $XXZ'$  and  $XXX'$  tops are their natural degenerations.

In section 8 we show that in process of obtaining trigonometric bosonisation formulas from elliptical ones (2)-(4) by tending  $\tau \rightarrow i\infty$  we obtain divergent results. To get finite expressions we perform regularization on the algebra  $\mathfrak{sl}(2, \mathbb{C})$  by introducing singular gauge transformation depending on  $\tau$ . We see that this transformation is the classical version of singular gauge transformation first introduced in [7], [8].

## 2 Tops on Lie algebras

Let  $\mathfrak{G}$  be a semisimple Lie algebra. The Euler-Arnold top on the algebra  $\mathfrak{G}$  is the Hamiltonian system on the coadjoint orbit:

$$M^{rot} = \{S \in \mathfrak{G}^* : S = g^{-1} S_0 g\}, \quad (6)$$

where  $g$  are elements of corresponding group  $g \in G$ . The phase space is equipped with a non-degenerate Kirillov-Kostant symplectic form:

$$w = \langle S_0, dgg^{-1} \wedge dgg^{-1} \rangle. \quad (7)$$

Where  $\langle, \rangle$  is the Killing form. Dynamics of the system is described by the Hamiltonian:

$$H_J = J_{ij} S_i S_j, \quad (8)$$

where  $J$  is some quadratic form on the algebra  $\mathfrak{G}$ . It is defined up to the quadratic Casimir element:

$$\Omega = \Omega_{ij} S_i S_j, \quad \{\Omega, A\} = 0 \quad \forall A \in \mathfrak{G}^*. \quad (9)$$

The Poisson structure on the algebra of functions can be described explicitly. On linear functions the Poisson brackets coincide with commutator:

$$\{A, B\} = [A, B], \quad A, B \in \mathfrak{G}^*,$$

whereupon Poisson brackets on polynomial functions can be evaluated by the Leibniz rule:

$$\{AB, C\} = \{A, C\}B + \{B, C\}A, \quad A, B, C \in F(M^{rot}).$$

Hence the equations of motion of the top in some basis  $S_i$ :

$$\frac{dS_k}{dt} = \{H_J, S_k\} = \{J_{ij} S_i S_j, S_k\} = 2J_{ij} C_{mjk} S_i S_m, \quad (10)$$

where  $C_{ijk}$  is the tensor of structure constants in this basis.

Hereby, the top is defined by choosing the quadratic form on the algebra. However two different quadratic form  $J_1$  and  $J_2$  can lead to the same equation of motion (10). If it is the case we will say that forms  $J_1$  and  $J_2$  are equivalent. More precisely

**Proposition.** *Let two quadratic forms are equivalent with respect to the action of a the automorphism group of the algebra  $\mathfrak{G}$ , i.e.  $J_{2ij} = J_{1km}T_{ki}T_{mj}$ , where  $T_{ij} \in \text{Aut}(\mathfrak{G})$ , then they define the same equations of motions. Prove. For  $H_{J_1}$  from (10) we have:*

$$\frac{dS_k}{dt} = 2J_{1ij}C_{mjk}S_iS_m, \quad (11)$$

for  $H_{J_2}$ :

$$\frac{dS'_k}{dt} = \{H_{J_2}, S'_k\} = \{J_{2lm}S_lS_m, S'_k\} = \{J_{1ij}T_{il}T_{jm}S_lS_m, S'_k\} = \{J_{1ij}S'_iS'_j, S'_k\},$$

where we denote  $S'_i = T_{ij}S^j$ . Therefore:

$$\frac{dS'_k}{dt} = 2J_{1ij}C'_{jkm}S'_iS'_m, \quad (12)$$

where  $C'_{ijk}$  is the tensor of structure constants in the basis  $S'_i$ . The action of the automorphism group preserves the commutation relations  $\{S'_i, S'_j\} = C_{ijk}S'_k$ , and hence  $C'_{ijk} = C_{ijk}$ . Hereby the equations (11) and (12) coincide.

It follows from above that the space of the quadratic forms on the algebra  $\mathfrak{G}$  fall into equivalence classes. The equivalent quadratic forms correspond to the equivalent tops.

### 3 $\text{sl}(2, \mathbb{C})$ -case

It follows from section 2 that the classification of the tops on the algebra  $\mathfrak{G}$  is reduced the to the classification of equivalence classes of quadratic forms on the algebra with respect to the action of the automorphism group. It turn out that in the case  $\mathfrak{G}=\text{sl}(2, \mathbb{C})$  the matrix of any quadratic form can be represented in one of the three canonical form. The canonical forms of matrix in some basis with the relations:

$$\{S_i, S_j\} = 2i\epsilon_{ijk}S_k, \quad (13)$$

are given by matrices:

$$XYZ) J = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}, \quad XXZ') J = \begin{bmatrix} \alpha & i\alpha & 0 \\ i\alpha & -\alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}, \quad XXX') J = \begin{bmatrix} \alpha & i\alpha & \beta \\ i\alpha & -\alpha & i\beta \\ \beta & i\beta & 0 \end{bmatrix}. \quad (14)$$

We will call bases of this kind the standard bases. In the standard basis (13) the Casimir function take the form:

$$\Omega = S_1^2 + S_2^2 + S_3^2. \quad (15)$$

The Casimir (15) is invariant with respect to the action of automorphism group. Thus, the matrix of  $A \in \text{Aut}(\text{sl}(2, \mathbb{C}))$  is orthogonal in the standard basis  $AA^t = 1$ . Therefore, the the matrix of quadratic form is transformed as the matrix of operator:

$$J' = AJA^t = AJA^{-1}.$$

So, the the canonical forms of the matrices of a quadratic forms on the algebra is exhausted by the symmetric matrices with different Jordan structures (14).

The Hamiltonians corresponding to these matrices have the forms:

$$XYZ) H = \alpha S_1^2 + \beta S_2^2 + \gamma S_3^2, \quad (16)$$

$$XXZ') H = \alpha(S_1 + iS_2)^2 + \beta S_3^2, \quad (17)$$

$$XXX') H = \alpha(S_1 + iS_2)^2 + \beta S_3(S_1 + iS_2); \quad (18)$$

It is useful to represent these Hamiltonians in the Chevalley basis:

$$XYZ) H = \alpha(e + if)^2 + \beta(e - if)^2 + \gamma h^2 \quad (19)$$

$$XXZ') H = \alpha e^2 + \beta h^2 \quad (20)$$

$$XXX') H = \alpha e^2 + \beta eh \quad (21)$$

$$\{h, e\} = 2e, \quad \{h, f\} = -2f, \quad \{e, f\} = h. \quad (22)$$

Therefore general  $\text{sl}(2, \mathbb{C})$ -top is described by one of the three Hamiltonians (19)-(21). In next sections we will investigate these Hamiltonians and find the corresponding two-particle systems. Thus, we present the complete connection between the  $\text{sl}(2, \mathbb{C})$ -tops and the systems of two particles.

## 4 Particles-Tops correspondence

In this section we describe the main technique for obtaining the bosonisation formulas. The system of two particles is described by the Hamiltonian:

$$H = p^2 + U(q). \quad (23)$$

It is defined on the two-dimensional phase space :

$$M^{part} = \{(p, q) \in \mathbb{C}^2\}, \quad w = dp \wedge dq. \quad (24)$$

with the canonical Poisson bracket:

$$\{f, g\} = \frac{\partial f}{\partial p} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial p}, \quad \{p, q\} = 1. \quad (25)$$

Consider a top with quadratic form  $J$ . To define the corresponding system of two particles we need explicit expressions for generators of the algebra  $S_i(p, q)$  in terms of  $p$  and  $q$ . With respect to the Poisson bracket (25) these functions obey the standard commutation relations (13):

$$\{S_i(p, q), S_j(p, q)\} = 2i\epsilon_{ijk}S_k(p, q)$$

Assume that there exist a map  $(p, q) \rightarrow S_k$  that transform the Hamiltonian (8) into (23). Then it is necessary to consider a linear in  $p$  ansatz for the generators of the algebra (the bosonisation formulas):

$$h = f_h(q)p + g_h(q), \quad e = f_e(q)p + g_e(q), \quad f = f_f(q)p + g_f(q), \quad (26)$$

where function  $f_h(q), f_e(q), f_f(q), g_h(q), g_e(q), g_f(q)$  should be defined. Commutation relations (22) lead to the system of differential equations:

$$f_h(q) \frac{df_f(q)}{dq} - f_f(q) \frac{df_h(q)}{dq} = -2f_f(q), \quad (27)$$

$$f_h(q) \frac{dg_f(q)}{dq} - f_f(q) \frac{dg_h(q)}{dq} = -2g_f(q), \quad (28)$$

$$f_h(q) \frac{df_e(q)}{dq} - f_e(q) \frac{df_h(q)}{dq} = 2f_e(q), \quad (29)$$

$$f_h(q) \frac{dg_e(q)}{dq} - f_e(q) \frac{dg_h(q)}{dq} = 2g_e(q), \quad (30)$$

$$f_e(q) \frac{df_f(q)}{dq} - f_f(q) \frac{df_e(q)}{dq} = f_h(q), \quad (31)$$

$$f_e(q) \frac{dg_f(q)}{dq} - f_f(q) \frac{dg_e(q)}{dq} = g_h(q). \quad (32)$$

Using (26) we rewrite the Hamiltonian of the top (8). It has the form of quadratic in  $p$  polynom:

$$H = \Lambda_1(f_h, f_e, f_f)p^2 + \Lambda_2(f_h, f_e, f_f, g_h, g_e, g_f)p + \Lambda_3(g_h, g_e, g_f), \quad (33)$$

where particular forms of the coefficients are defined by  $J$ . By comparing (33) with (23) we find:

$$\Lambda_1(f_h, f_e, f_f) = 1, \quad (34)$$

$$\Lambda_2(f_h, f_e, f_f, g_h, g_e, g_f) = 0. \quad (35)$$

There is an additional condition, because the Casimir element is a constant on orbits:

$$h^2 + 4ef = \nu^2. \quad (36)$$

The last equations (34)-(36) play the role of boundary conditions for the system (27)-(32) and allow us to solve this system explicitly. As a result we obtain the function  $\Lambda_3(q)$  that play the role of the interaction potential.

## 5 The $XXX'$ -case

Let us consider the system of two particles corresponding to  $XXX'$  top following the receipt of the previous section. According with (21) the Hamiltonian of top in  $XXX'$ -case has the following form:

$$H = \alpha e^2 + \beta eh.$$

where we assume  $\alpha = 1$  for normalization:

$$H = e^2 + \beta eh. \quad (37)$$

Substituting (26) into (37) we find the following boundary conditions:

$$\Lambda_1(f_e, f_h, f_f) = f_e^2(q) + \beta f_e(q) f_h(q) = 1, \quad (38)$$

$$\Lambda_2(f_e, f_h, f_f, g_e, g_h, g_f) = 2f_e(q)g_e(q) + \beta f_e(q)g_h(q) + \beta f_h(q)g_e(q) = 0. \quad (39)$$

**Proposition.** *The solution of the system (27)-(32) satisfying the boundary conditions (38), (39), and (36) has the following form:*

$$f_h(q) = \frac{1 - q^2 \beta^2}{q \beta^2}, \quad (40)$$

$$g_h(q) = \frac{\nu(q^2 \beta^2 + 1)}{2q^2 \beta^2}. \quad (41)$$

$$f_e(q) = -\frac{1}{q \beta}, \quad (42)$$

$$g_e(q) = -\frac{\nu}{2q^2 \beta}, \quad (43)$$

$$f_f(q) = \frac{(1 - q^2 \beta^2)^2}{4q \beta^3}, \quad (44)$$

$$g_f(q) = \frac{\nu(1 - q^2 \beta^2)(1 + 3q^2 \beta^2)}{8q^2 \beta^3}. \quad (45)$$

Hereby the bosonisation formulas have the form:

$$e = -\frac{p}{q \beta} - \frac{\nu}{2q^2 \beta}, \quad (46)$$

$$h = \frac{1 - q^2 \beta^2}{q \beta^2} p + \frac{\nu(q^2 \beta^2 + 1)}{2q^2 \beta^2}, \quad (47)$$

$$f = \frac{(1 - q^2 \beta^2)^2}{4q \beta^3} p + \frac{\nu(1 - q^2 \beta^2)(1 + 3q^2 \beta^2)}{8q^2 \beta^3}. \quad (48)$$

For the Hamiltonian we have:

$$H = e^2 + \beta eh = p^2 - \frac{\nu^2}{(2q)^2}. \quad (49)$$

This Hamiltonian describe the two-particle Calogero-Moser system with a rational potential. Thus, we have proved that  $XXX'$  top is equivalent to the rational Calogero-Moser system.

## 6 The $XXZ'$ -case

Let us repeat previous calculations for  $XXZ'$  top. According with (20) in this case the Hamiltonian of top has the form:

$$H = e^2 + \gamma^2 h^2,$$

where we assume  $\alpha = 1$ ,  $\beta = \gamma^2$  for simplicity. The system of the boundary conditions (34), (35) in this case have the form:

$$\Lambda_1(f_e, f_h, f_f) = f_e^2(q) + \gamma^2 f_h^2(q) = 1, \quad (50)$$

$$\Lambda_2(f_e, f_h, f_f, g_e, g_h, g_f) = f_e(q)g_e(q) + \gamma^2 f_h(q)g_h(q) = 0. \quad (51)$$

**Proposition.** *Solution of the system (27)-(32) satisfying the boundary conditions (50),(51), and (36) have the following form:*

$$f_h(q) = -\frac{1}{\gamma th(2\gamma q)}, \quad (52)$$

$$g_h(q) = -\frac{\nu}{sh^2(2\gamma q)}, \quad (53)$$

$$f_e(q) = -\frac{i}{sh(2\gamma q)}, \quad (54)$$

$$g_e(q) = -\frac{i\nu\gamma ch(2\gamma q)}{sh^2(2\gamma q)}, \quad (55)$$

$$f_f(q) = -\frac{ich^2(2\gamma q)}{4\gamma^2 sh(2\gamma q)}, \quad (56)$$

$$g_f(q) = \frac{i\nu ch(2\gamma q)(ch^2(2\gamma q) - 2)}{4\gamma sh^2(2\gamma q)}. \quad (57)$$

The bosonisation formulas (26) in  $XXZ'$ -case take the form:

$$e = -\frac{i}{sh(2\gamma q)}p - \frac{i\nu\gamma ch(2\gamma q)}{sh^2(2\gamma q)}, \quad (58)$$

$$h = -\frac{1}{\gamma th(2\gamma q)}p - \frac{\nu}{sh^2(2\gamma q)}, \quad (59)$$

$$f = -\frac{ich^2(2\gamma q)}{4\gamma^2 sh(2\gamma q)}p + \frac{i\nu ch(2\gamma q)(ch^2(2\gamma q) - 2)}{4\gamma sh^2(2\gamma q)}. \quad (60)$$

We obtain the following expression for the Hamiltonian:

$$H = e^2 + \gamma^2 h^2 = p^2 - \frac{\gamma^2 \nu^2}{sh^2(2\gamma q)}. \quad (61)$$

It corresponds to two-particle Calogero-Moser system with the trigonometric potential. Thus, we have proved the equivalence of the  $XXZ'$ -top and the trigonometric Calogero-Moser system for two particles.

## 7 The $XYZ$ -case

According with (19) the Hamiltonian of  $XYZ$  top is given by the expression:

$$H = \alpha(e + f)^2 + \beta(e - f)^2 + \gamma h^2.$$

Let us consider the Casimir function:

$$\Omega = h^2 + 4ef = h^2 + (e + f)^2 - (e - f)^2,$$

If we subtract  $\gamma\Omega$  from the Hamiltonian, and assume  $\beta + \gamma = k^2, \gamma - \alpha = 1$  for normalization we find:

$$H = k^2(e - f)^2 - (e + f)^2. \quad (62)$$

The boundary conditions (27)-(32) for this Hamiltonian take the form:

$$\Lambda_2 = k^2(f_e(q) - f_f(q))(g_e(q) - g_f(q)) - (f_e(q) + f_f(q))(g_e(q) + g_f(q)) = 0, \quad (63)$$

$$\Lambda_1 = k^2(f_e(q) - f_f(q))^2 - (f_e(q) + f_f(q))^2 = 1. \quad (64)$$

**Proposition.** *The solution of the system (27)-(32) satisfying the boundary conditions (63),(64), and (36) has the following form:*

$$f_h = \frac{1}{sn(2q, k)k}, \quad (65)$$

$$g_h = -\frac{\nu dn(2q, k) cn(2q, k)}{sn(2q, k)^2 k}, \quad (66)$$

$$f_f = -1/2 \frac{k cn(2q, k) + dn(2q, k)}{sn(2q, k)k\sqrt{1-k^2}}, \quad (67)$$

$$g_f = 1/2 \frac{\nu (k dn(2q, k) + cn(2q, k))}{sn(2q, k)^2 k\sqrt{1-k^2}}, \quad (68)$$

$$f_e = -1/2 \frac{k cn(2q, k) - dn(2q, k)}{sn(2q, k)k\sqrt{1-k^2}}, \quad (69)$$

$$g_e = 1/2 \frac{k\nu dn(2q, k) - \nu cn(2q, k)}{sn(2q, k)^2 k\sqrt{1-k^2}}, \quad (70)$$

where  $dn(q, k)$ ,  $sn(q, k)$ ,  $cn(q, k)$  are the elliptic Jacobi functions with the modular parameter  $k$ . The bosonisation formulas have the form:

$$e = -1/2 \frac{(k cn(2q, k) - dn(2q, k))p}{sn(2q, k)k\sqrt{1-k^2}} + 1/2 \frac{k\nu dn(2q, k) - \nu cn(2q, k)}{sn(2q, k)^2 k\sqrt{1-k^2}}, \quad (71)$$

$$f = -1/2 \frac{(k cn(2q, k) + dn(2q, k))p}{sn(2q, k)k\sqrt{1-k^2}} + 1/2 \frac{\nu (k dn(2q, k) + cn(2q, k))}{sn(2q, k)^2 k\sqrt{1-k^2}}, \quad (72)$$

$$h = \frac{p}{sn(2q, k)k} - \frac{\nu dn(2q, k) cn(2q, k)}{sn(2q, k)^2 k}. \quad (73)$$

It is more useful to present the bosonisation formulas in the standard basis  $S_1, S_2, S_3$ :

$$S_1 = e + f, \quad S_2 = i(f - e), \quad S_3 = h, \quad \{S_i, S_j\} = 2i\epsilon_{ijk}S_k. \quad (74)$$

By expressing the Jacobi functions in terms of the elliptic theta functions we obtain the following result:

$$S_1 = \frac{1}{\sqrt{1-k^2}} \left( -\frac{cn(2q, k)}{sn(2q, k)}p + \frac{\nu dn(2q, k)}{sn(2q, k)^2} \right) = -\frac{\theta_{10}(0)\theta_{10}(2q)}{\vartheta'(0)\vartheta(2q)}p + \frac{\theta_{10}^2(0)\theta_{00}(2q)\theta_{01}(2q)}{\theta_{00}(0)\theta_{01}(0)\vartheta^2(2q)}\nu, \quad (75)$$

$$S_2 = \frac{1}{ik\sqrt{1-k^2}} \left( \frac{dn(2q, k)}{sn(2q, k)}p - \frac{\nu cn(2q, k)}{sn(2q, k)^2} \right) = \frac{\theta_{00}(0)\theta_{00}(2q)}{i\vartheta'(0)\vartheta(2q)}p - \frac{\theta_{00}^2(0)\theta_{10}(2q)\theta_{01}(2q)}{i\theta_{10}(0)\theta_{01}(0)\vartheta^2(2q)}\nu, \quad (76)$$

$$S_3 = \frac{1}{k} \left( \frac{p}{sn(2q, k)} - \frac{\nu dn(2q, k) cn(2q, k)}{sn(2q, k)^2} \right) = -\frac{\theta_{01}(0)\theta_{01}(2q)}{\vartheta'(0)\vartheta(2q)}p + \frac{\theta_{01}^2(0)\theta_{00}(2q)\theta_{10}(2q)}{\theta_{00}(0)\theta_{10}(0)\vartheta^2(2q)}\nu, \quad (77)$$

where  $\vartheta(q)$ ,  $\theta_{\alpha\beta}(q)$  are the standard theta function and theta function with characteristics respectively. For the Hamiltonian function we have:

$$H = k^2(e - f)^2 - (e + f)^2 = p^2 - \nu^2 \wp(2q, k), \quad (78)$$

where  $\wp(q, k)$  is the Weierstrass elliptic function. The presented Hamiltonian is the Hamiltonian of the elliptic Calogero-Moser system. It is worth noticing that the presented formulas coincide with the formulas obtained in [1].

## 8 Connection between XYZ and XXZ' tops

In this section we explain the interrelation between the bosonisation formulas corresponding to XYZ and XXZ' tops. In the limit  $k \rightarrow 0$  the elliptic Weierstrass function behave as:

$$\wp(q, k) \rightarrow \frac{1}{\sin^2(q)},$$

and the Hamiltonian of elliptic the Calogero-Moser system is transformed into the trigonometric one. Thus, it is natural to suppose that in this limit  $XYZ$  top transform into  $XXZ'$  top. However, the matrices of quadratic forms defining these tops are not transformed into each other in this way. To understand what is really going on let us consider the behavior of the bosonisation formulas in the limit  $k \rightarrow 0$ . From equations (75)-(77) we can find:

$$S_1 \rightarrow -p \frac{\cos(2q)}{\sin(2q)} + \nu \frac{1}{\sin^2(2q)} \quad (79)$$

$$S_2 \rightarrow \frac{1}{ik} \Phi(q, p), \quad (80)$$

$$S_3 \rightarrow \frac{1}{k} \Phi(q, p), \quad (81)$$

where

$$\Phi(q, p) = p \frac{1}{\sin(2q)} - \nu \frac{\cos(2q)}{\sin^2(2q)}. \quad (82)$$

Thus, in this limit the bosonisation formulas diverge. To obtain a finite expressions we need some regularizing procedure. Let us consider the behavior of Chevalley generators of the algebra in this limit:

$$h = S_1, \quad e = \frac{1}{2}(S_3 - i S_2) \sim k, \quad f = \frac{1}{2}(S_3 + i S_2) \sim \frac{1}{k} \quad (83)$$

Let us notice that the transformation  $h' \rightarrow h, \quad e' \rightarrow \frac{1}{k}e, \quad f' \rightarrow kf$  does not change the commutation relations, and we have well defined limit  $k \rightarrow 0$ . After turning back to the standard basis we find the following expressions:

$$S'_1 = h' = -p \frac{\cos(2q)}{4 \sin(2q)} + \nu \frac{1}{\sin^2(2q)}, \quad (84)$$

$$S'_2 = \frac{1}{i}(f' - e') = \frac{1}{4i} \left( p \frac{4 + \cos(2u)}{\sin(2q)} + \frac{\nu \cos(2q)(5 + \sin^2(2q))}{\sin^2 2q} \right), \quad (85)$$

$$S'_3 = (e' + f') = \frac{1}{4} \left( p \frac{4 - \cos^2(2q)}{\sin(2q)} - \frac{\nu \cos(2q)(3 - \sin^2(2q))}{\sin^2(2q)} \right). \quad (86)$$

It is easy to check that these expressions coincide with (58)-(60) at some value of constant  $\gamma$ . They have correct commutation relations:

$$\{S'_i, S'_j\} = 2i\epsilon_{ijk} S'_k, \quad (87)$$

$$(S'_1)^2 + (S'_2)^2 + (S'_3)^2 = \nu^2. \quad (88)$$

In the new basis the matrix of the quadratic form is given by  $J' = T J T^t$ , where  $T = A^{-1} B A$  and

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 i & 1/2 \\ 0 & 1/2 i & -1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k^{-1} & 0 \\ 0 & 0 & k \end{bmatrix} \quad (89)$$

We see that matrix  $T$  is singular at the point  $k = 0$ , but its determinant is equal to one for any value of parameter  $k$ . On calculating the matrix  $J'$  in the new basis we see that its form correspond to  $XXZ'$  case. Hereby  $XYZ$  and  $XXZ'$  tops are connected by some singular gauge transformation, which depends on modular parameter  $k$ . Transformations of that kind were introduced in [7],[8]. In this articles new solutions of Yang-Baxter equation for algebras of  $A_n$  type were obtained. These solutions present non-dynamical R-matrices for quantum trigonometric Calogero-Moser system. These new trigonometric R-matrices were obtained from elliptic ones by applying to them some singular gauge transformation which depends on modular parameter  $k$  and singular at  $k = 0$ . Thus the singular gauge transformation introduced in this section is its classical version.

## 9 Conclusions

Let us sum up shortly main results of the paper. We have demonstrated that the  $\mathfrak{sl}(2, \mathbb{C})$ -tops are equivalent to the two-particle systems of Calogero-Moser type. We proved that a Hamiltonian of generic top is equivalent



to one of the three canonical Hamiltonians. We denoted these Hamiltonians as  $XYZ$ ,  $XXZ'$ , and  $XXX'$  respectively. For each canonical Hamiltonian we found natural Darboux coordinates on coadjoint orbits (the bosonisation formulas). In terms of these coordinates dynamics of  $XYZ$ ,  $XXZ'$ , and  $XXX'$  tops is presented as dynamics of elliptical, trigonometrical or rational two-particle Calogero-Moser system respectively. In this way we showed the equivalence between the  $\mathfrak{sl}(2, \mathbb{C})$ -tops and the two-particle Calogero-Moser systems explicitly.

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